

Notes on the Fourier Transform

The Fourier transform is a mathematical method for describing a continuous function as a series of sine and cosine functions. The Fourier Transform is produced by applying a series of "Test Frequencies" to determine the contribution of each test function to the observed signal. This document will outline the fundamentals of how the Fourier Transform (FT) works. For additional details, a series of mathcad documents designed as homework exercises are available.

As an example, start with a 2 Hz cosine signal with an amplitude of 1. This signal is "aquired" by a computer using an A-D (analog to digital) board in the computer. For this experiment the computer samples the signal once every 1.953 ms for a total of 1 second. This is a sampling rate of 512 Hz and it gives a data file with 512 points.

Sample and signal parameters:

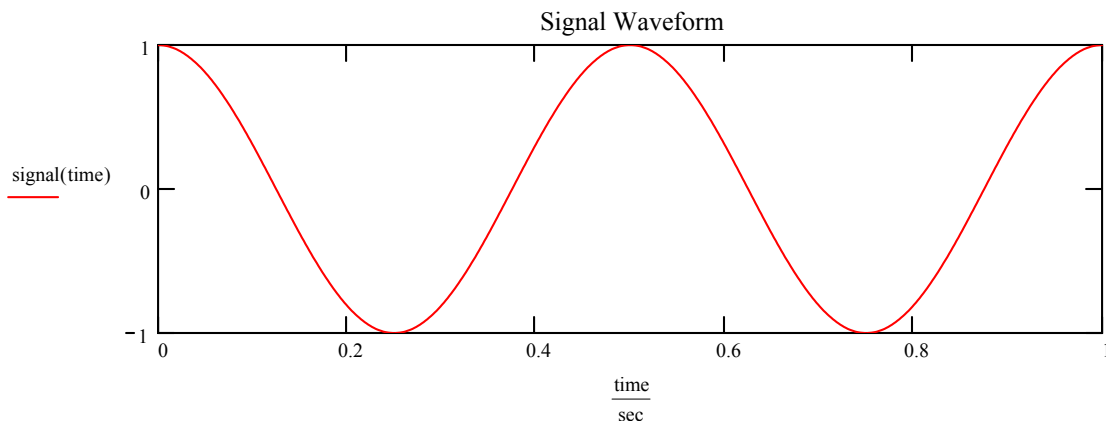
| | | |
|---|---|---|
| Signal frequency in Hz. | $v_{\text{signal}} := (2 \cdot \text{Hz})$ | |
| Signal frequency in radians per second. | $\omega_{\text{signal}} := 2 \cdot \pi \cdot v_{\text{signal}}$ | |
| Signal amplitude | $A := 1$ | |
| Time between points (Dwell Time) | $DT := 1.953 \cdot \text{ms}$ | $\text{ms} := 10^{-3} \cdot \text{sec}$ |
| Total acquisition time | $t_{\text{acquire}} := 1 \cdot \text{sec}$ | |

Calculated Values from Above:

| | |
|---|---|
| Sampling Rate | $\text{rate} := \frac{1}{DT}$ |
| | $\text{rate} = 512.033 \text{ Hz}$ |
| Number of points (floor is a function that returns an integer, used since you can't have a fraction of a measurement) | $N := \text{floor}(\text{rate} \cdot t_{\text{acquire}})$ |
| | $N = 512$ |

Equation for the signal at time t and to define the time of each point.

$$\text{signal}(t) := A \cdot \cos(\omega_{\text{signal}} \cdot t) \quad \text{time} := 0, \frac{1}{N} \cdot t_{\text{acquire}} \dots t_{\text{acquire}}$$



This is just a cosine wave with an amplitude of 1 and a frequency of 2 Hz. notice that there are two complete cycles in the 1 second that data is aquired. Remember this curve is made from 512 points.

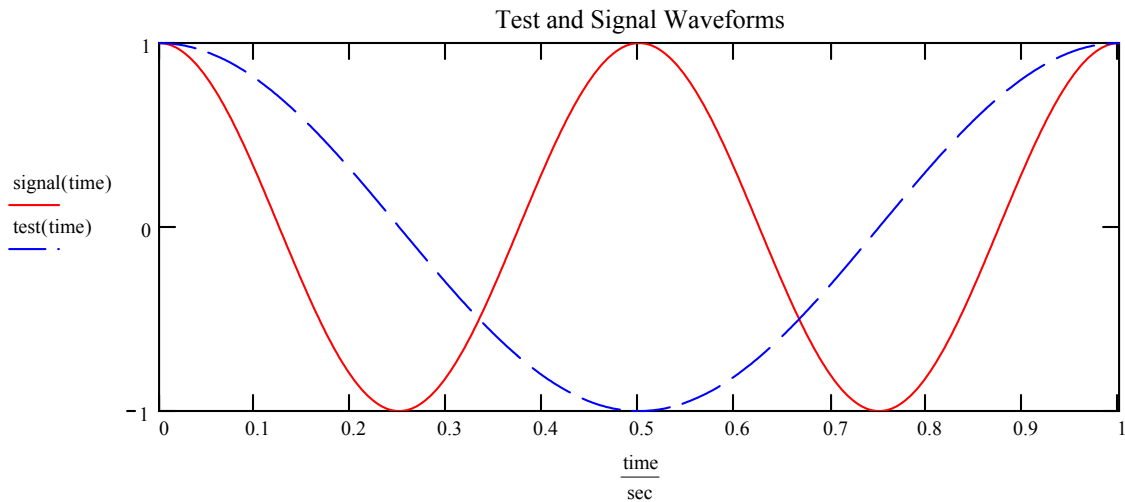
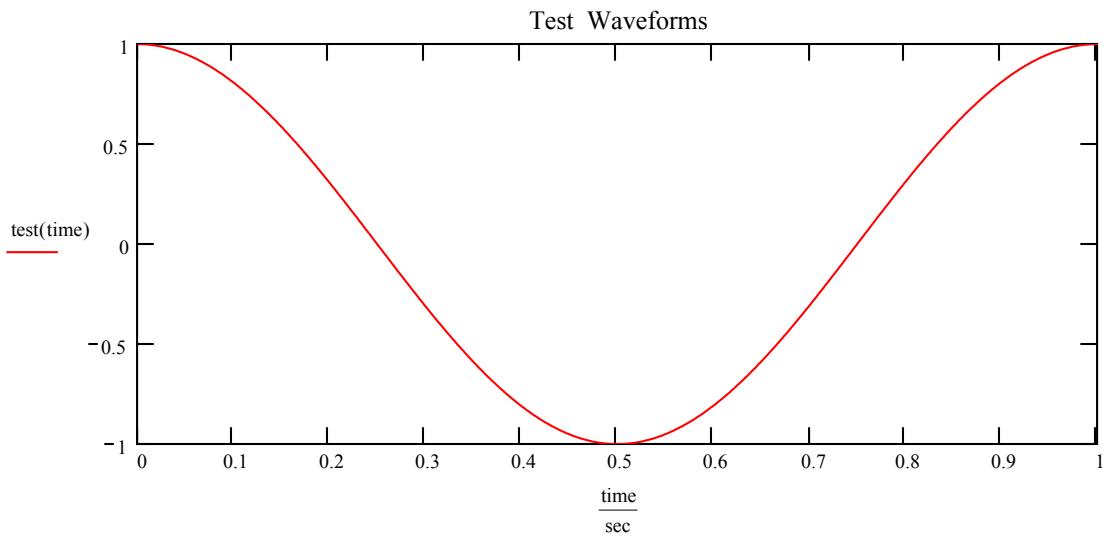
Now that we have a signal, the fourier transform can determine the frequency components in the signal. This is done by applying a series of test frequencies. For each test frequency, the signal is multiplied by the test frequency to produce a new waveform. The integration of the new (product) waveform is the signal at that test frequency

First let's use a 1 Hz test frequency to determine the intensity of the 1 Hz component in the signal.

$$v_{\text{test}} := 1 \cdot \text{Hz} \qquad \omega_{\text{test}} := 2 \cdot \pi \cdot v_{\text{test}}$$

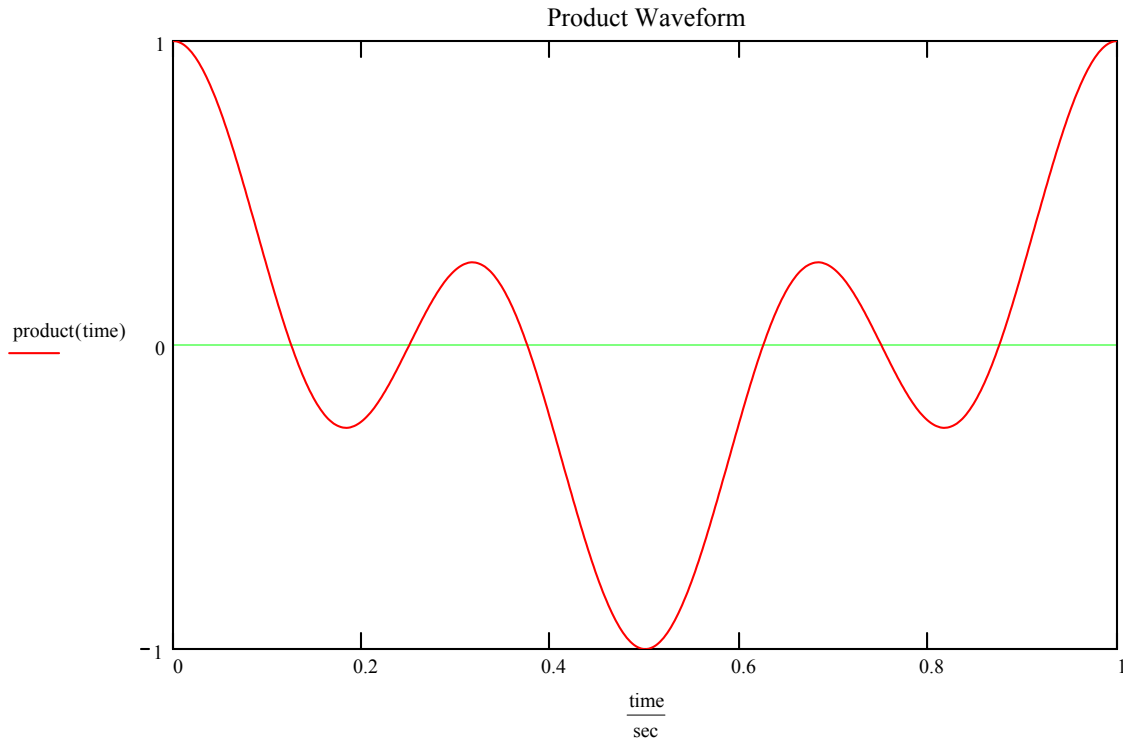
Generate the test wave:

$$\text{test}(t) := \cos(\omega_{\text{test}} \cdot t)$$



Next the product wave is produced by multiplying the two waveforms (signal and test)

Multiply the two waveforms, point by point: $\text{product}(t) := \text{test}(t) \cdot \text{signal}(t)$



Integrate the product function to determine the area of the wave. This is the intensity of the signal at 1 Hz (the test frequency):

Analytically (with calculus)

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}(t) dt = 0 \text{ sec}$$

Numerically (add the points)

$$\sum_{i=0}^{N-1} \frac{\text{product}\left(i \cdot \frac{t_{\text{acquire}}}{N}\right)}{N} = 0$$

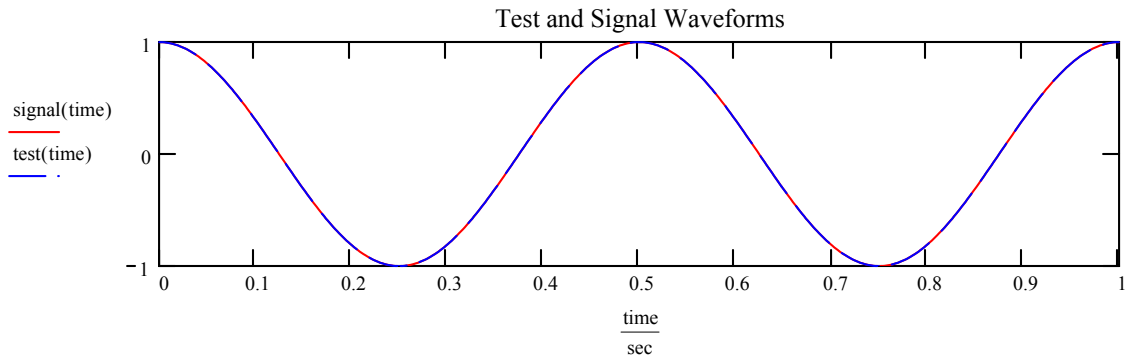
Also integrate the function by looking closely at it. Notice that all the positive areas (above 0) and the negative areas (below zero) cancel. This result should make sense. The test frequency was 1 Hz, but the signal was 2 Hz. There is no signal at 1 Hz.

Now let's try it again with a 2 Hz test frequency. (What do you expect the answer to be?)

$$v_{\text{test}} := 2 \cdot \text{Hz} \qquad \omega_{\text{test}} := 2 \cdot \pi \cdot v_{\text{test}}$$

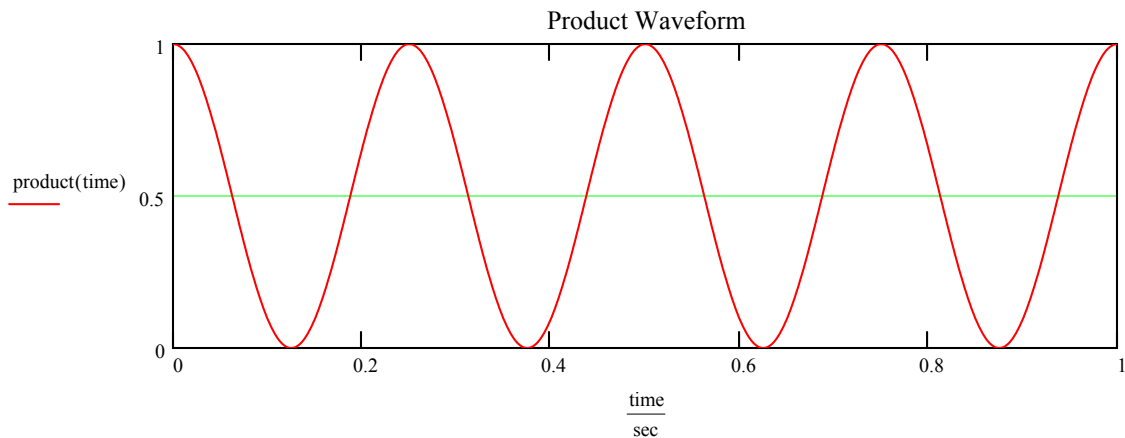
Generate the test wave:

$$\text{test}(t) := \cos(\omega_{\text{test}} \cdot t)$$



Next the product wave is produced by multiplying the two waveforms (signal and test)

Multiply the two waveforms, point by point: $\text{product}(t) := \text{test}(t) \cdot \text{signal}(t)$



Integrate the product function to determine the area of the wave. This is the intensity of the signal at 2 Hz (the test frequency):

Analytically (with calculus)

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}(t) dt = 0.5 \text{ sec}$$

Numerically (add the points)

$$\sum_{i=0}^{N-1} \frac{\text{product}\left(i \cdot \frac{t_{\text{acquire}}}{N}\right)}{N} = 0.5$$

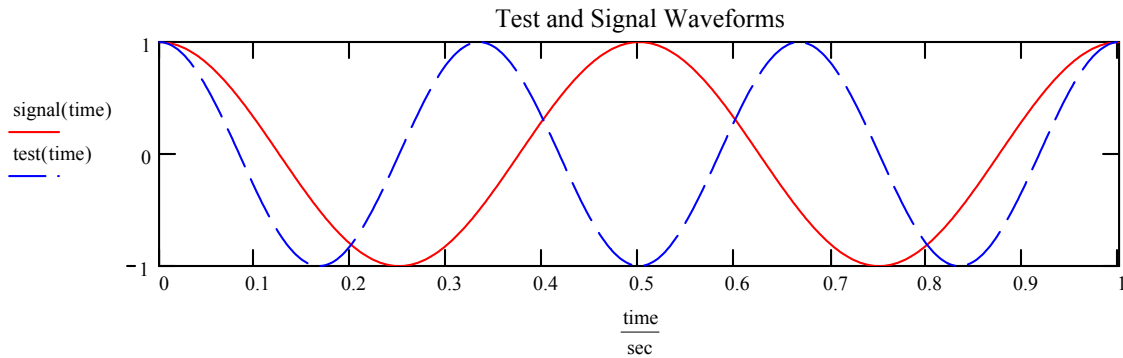
Also integrate the function by looking closely at it. Notice that the entire wave is above 0 so nothing cancels this time. The product waveform integrates to a positive value. This should make sense since the signal and the test frequency are both 2 Hz.

Finally let's try it again with a 3 Hz test frequency. (What do you expect the answer to be?)

$$v_{\text{test}} := 3 \cdot \text{Hz} \qquad \omega_{\text{test}} := 2 \cdot \pi \cdot v_{\text{test}}$$

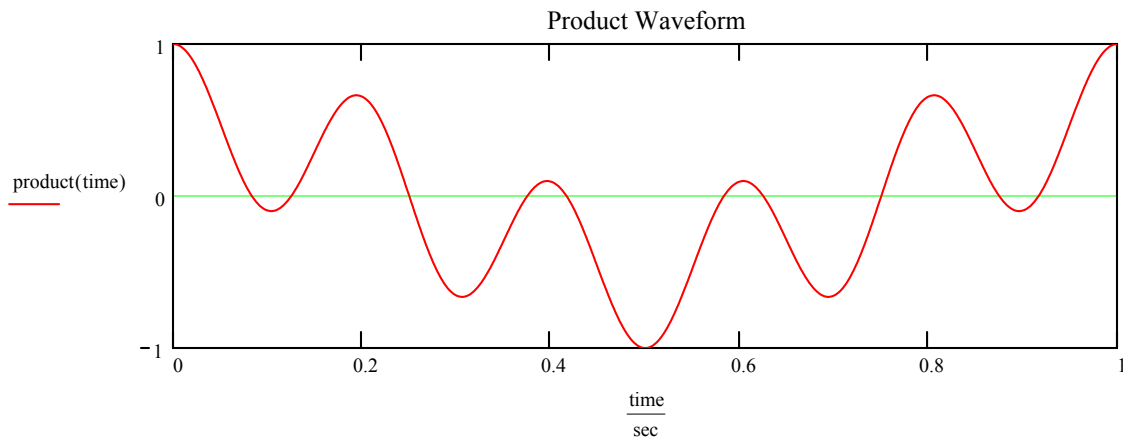
Generate the test wave:

$$\text{test}(t) := \cos(\omega_{\text{test}} \cdot t)$$



Next the product wave is produced by multiplying the two waveforms (signal and test)

Multiply the two waveforms, point by point: $\text{product}(t) := \text{test}(t) \cdot \text{signal}(t)$



Integrate the product function to determine the area of the wave. This is the intensity of the signal at 3 Hz (the test frequency):

Analytically (with calculus)

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}(t) dt = 0 \text{ sec}$$

Numerically (add the points)

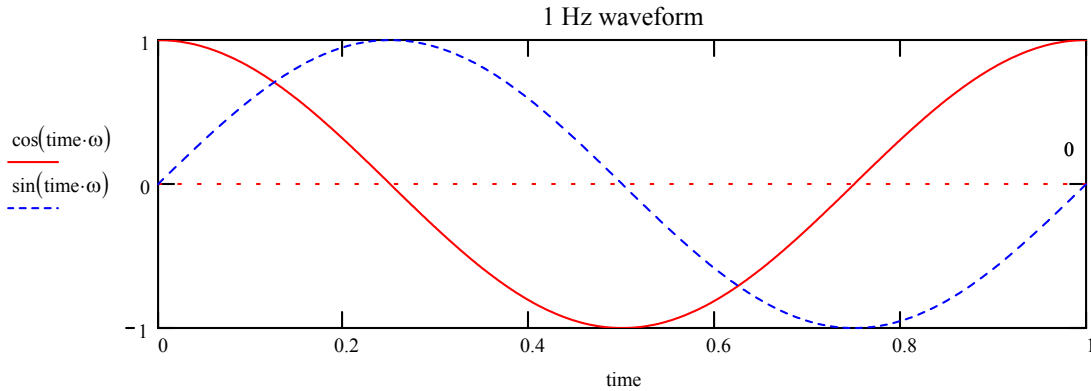
$$\sum_{i=0}^{N-1} \frac{\text{product}\left(i \cdot \frac{t_{\text{acquire}}}{N}\right)}{N} = 0$$

Also integrate the function by looking closely at it. Notice that once again the negative and positive portions of the waveform all cancel. This should make sense since the signal is at 2 Hz and the test frequency is 3 Hz. In a "real" experiment the signal will contain MANY different frequencies. All at different intensities. The FT can handle this just fine.

Next we will examine the idea of "phase". The original "theory" of the FT is that any continuous function may be described as a series of cosine and sine waves. The cosine part is called the "real" and the sine part is called the "imaginary". This is analogous to real and imaginary numbers used in mathematics. This concept will be very important for FT-NMR so a bit of detail is relevant here. (Note it is also important in FT-IR, but you do not usually realize it).

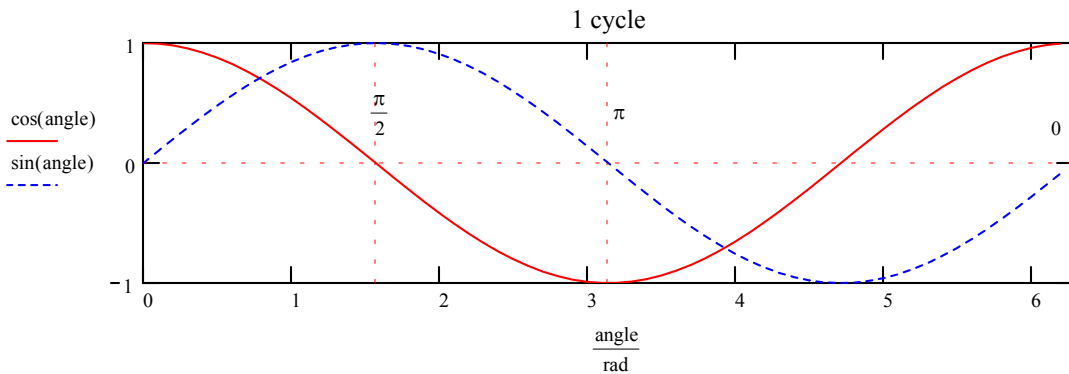
A 1 Hz cosine and sine waves looks like this:

$$\nu := 1 \cdot \text{Hz} \quad \omega := 2 \cdot \pi \cdot \nu$$

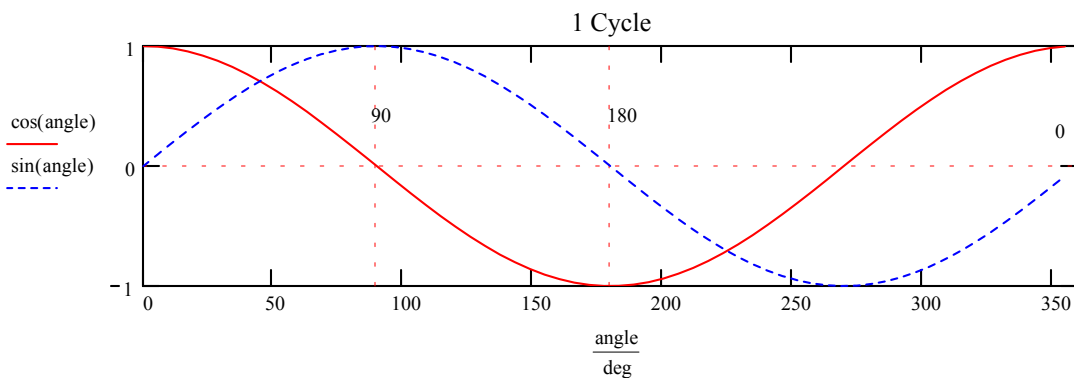


The cosine wave may also be shown vs the angle (A 1 Hz wave just goes around the circle once each second). This angle is in radians or degrees.

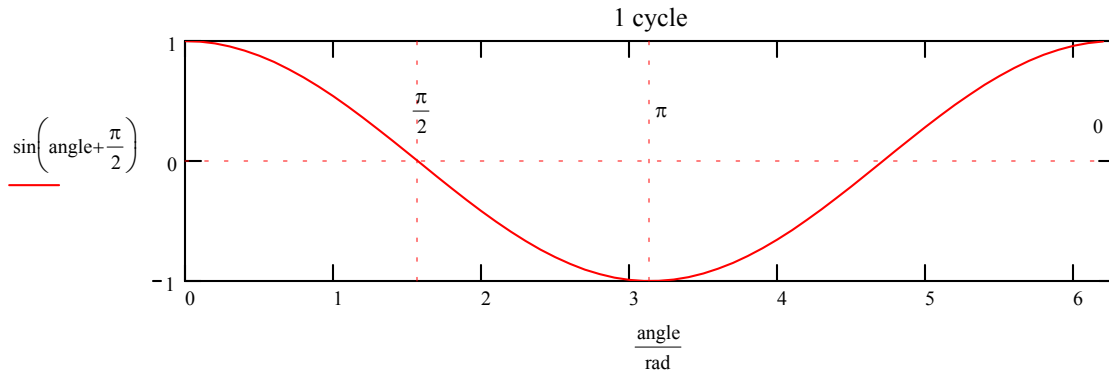
$$\text{angle} := 0, 0.1 \dots 2 \cdot \pi$$



The wave may also be displayed with the angle in degrees.



Notice that the sine and cosine waves are separated by a 90 degree (or $\pi/2$ radian) "phase angle". Watch what happens to a sine wave if a $\pi/2$ degree phase angle is added to it.



Now it is the same as a cosine wave. So a sine and cosine wave are separated by 90 degrees. If you subtract 90 degrees from a cosine wave it will look just like a sine wave.

Next we can relate this to the phase of a signal waveform. If the signal waveform is a pure cosine wave (like the signal we used above). The signal is "real". Alternatively, if the signal is a pure sine wave it is an "imaginary" signal. NOTE: *These are just names, they do not imply anything about the "reality" of the signal.*

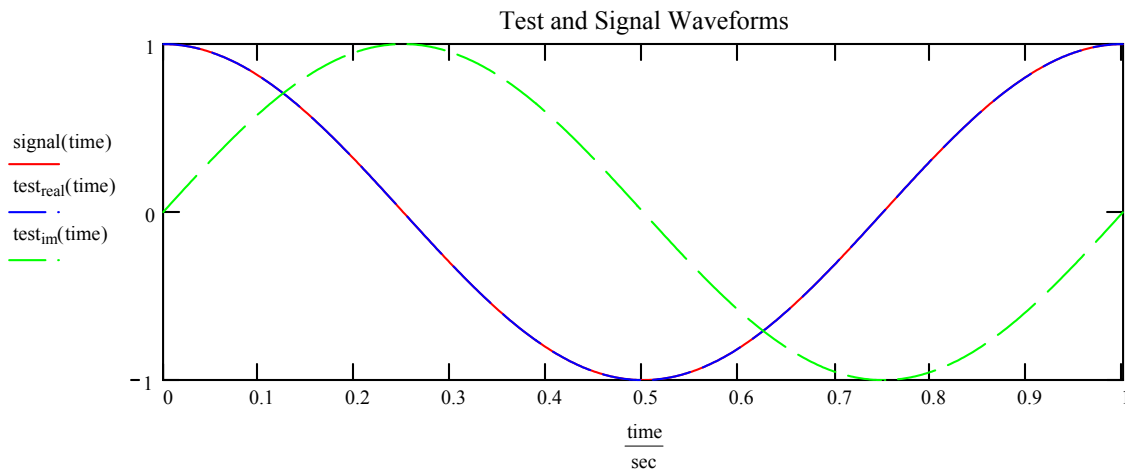
We can show this by now using two test waves. One cosine and one sine. Watch what happens when we look at a 1 Hz "real" signal, with 1 Hz cosine and 1 Hz sine test waves.

Generate the waveforms:

$$\text{signal}(t) := \cos(2 \cdot \pi \cdot \text{Hz} \cdot t)$$

$$\text{test}_{\text{real}}(t) := \cos(2 \cdot \pi \cdot \text{Hz} \cdot t)$$

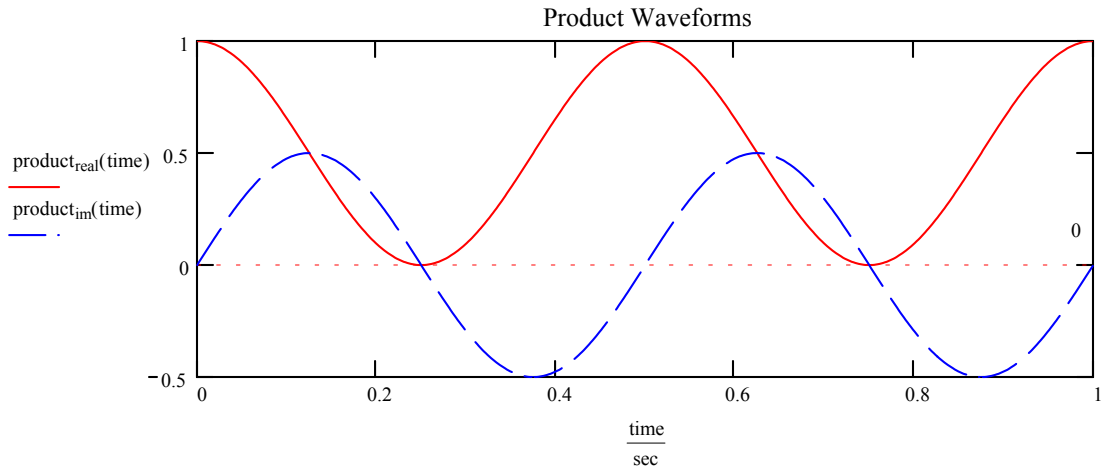
$$\text{test}_{\text{im}}(t) := \sin(2 \cdot \pi \cdot \text{Hz} \cdot t)$$



Next the product wave is produced by multiplying the two waveforms (signal and test)

$$\text{product}_{\text{real}}(t) := \text{test}_{\text{real}}(t) \cdot \text{signal}(t)$$

$$\text{product}_{\text{im}}(t) := \text{test}_{\text{im}}(t) \cdot \text{signal}(t)$$



Integrate the product function to determine the "real" and "imaginary" signal.

Integrate the real signal:

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}_{\text{real}}(t) dt = 0.5 \text{ sec}$$

Integrate the imaginary signal:

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}_{\text{im}}(t) dt = 0 \text{ sec}$$

This result is consistent with what you expect. Since the signal is a pure cosine wave it should be all "real". It also is consistent with visual inspection of the product waveforms.

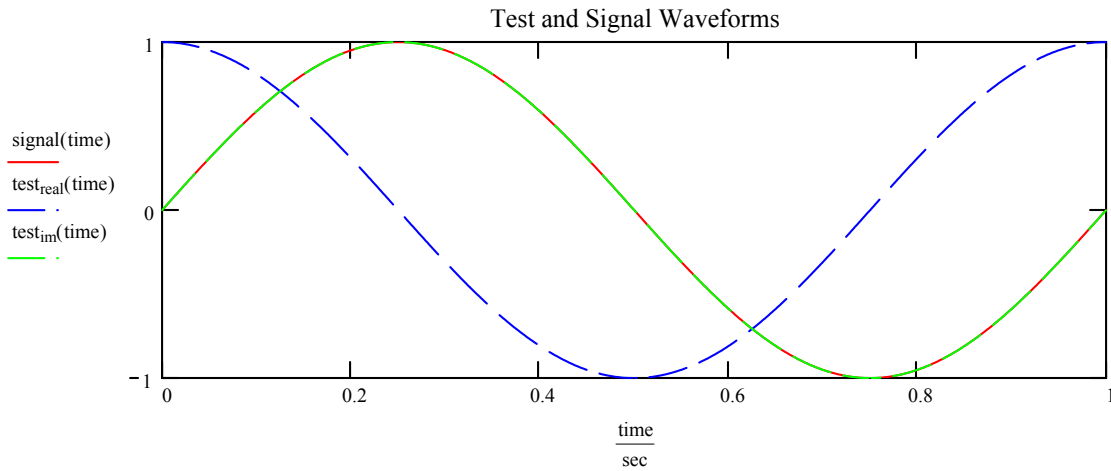
Next let's see what happens if the signal is a pure sine wave.

Generate the waveforms:

$$\text{signal}(t) := \sin(2 \cdot \pi \cdot \text{Hz} \cdot t)$$

$$\text{test}_{\text{real}}(t) := \cos(2 \cdot \pi \cdot \text{Hz} \cdot t)$$

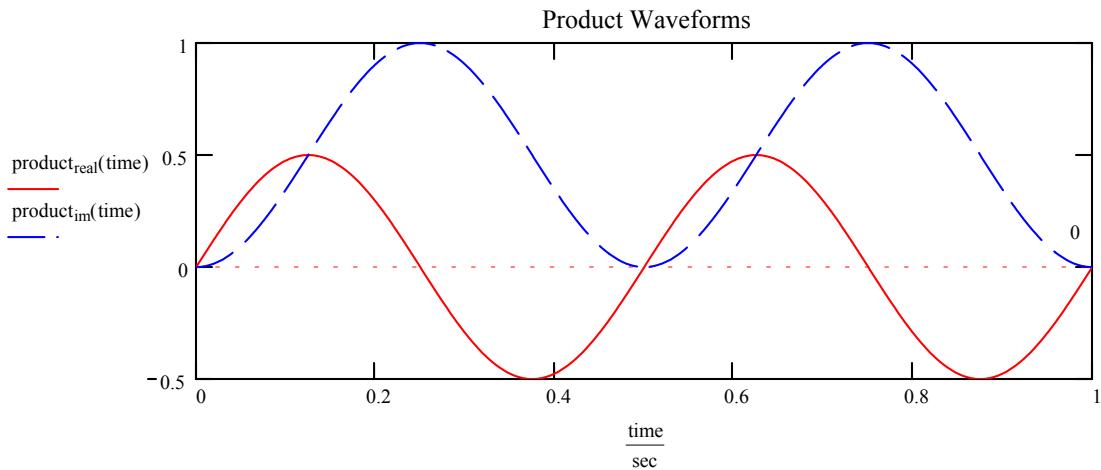
$$\text{test}_{\text{im}}(t) := \sin(2 \cdot \pi \cdot \text{Hz} \cdot t)$$



Next the product wave is produced by multiplying the two waveforms (signal and test)

$$\text{product}_{\text{real}}(t) := \text{test}_{\text{real}}(t) \cdot \text{signal}(t)$$

$$\text{product}_{\text{im}}(t) := \text{test}_{\text{im}}(t) \cdot \text{signal}(t)$$



Integrate the product function to determine the "real" and "imaginary" signal.

Integrate the real signal:

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}_{\text{real}}(t) dt = 0 \text{ sec}$$

Integrate the imaginary signal:

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}_{\text{im}}(t) dt = 0.5 \text{ sec}$$

This result is consistent with what you expect. Since the signal is a pure sine wave it should be all "imaginary". It also is consistent with visual inspection of the product waveforms.

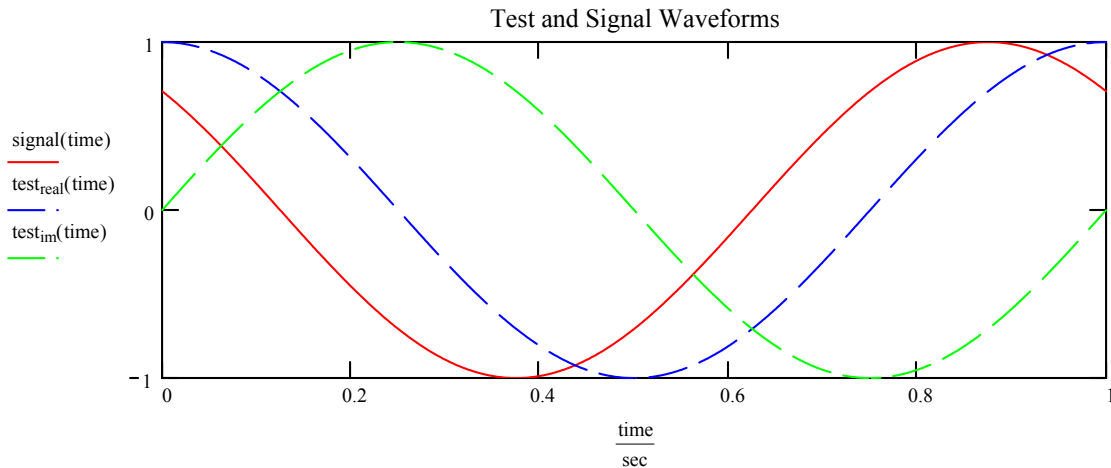
Next what happens if the signal has a phase between 0 (cosine or real) and 90 (sine or imaginary).

Generate the waveforms:

$$\text{signal}(t) := \cos\left(2 \cdot \pi \cdot \text{Hz} \cdot t + \frac{\pi}{4}\right) \quad \text{A 45 degree phase angle}$$

$$\text{test}_{\text{real}}(t) := \cos(2 \cdot \pi \cdot \text{Hz} \cdot t)$$

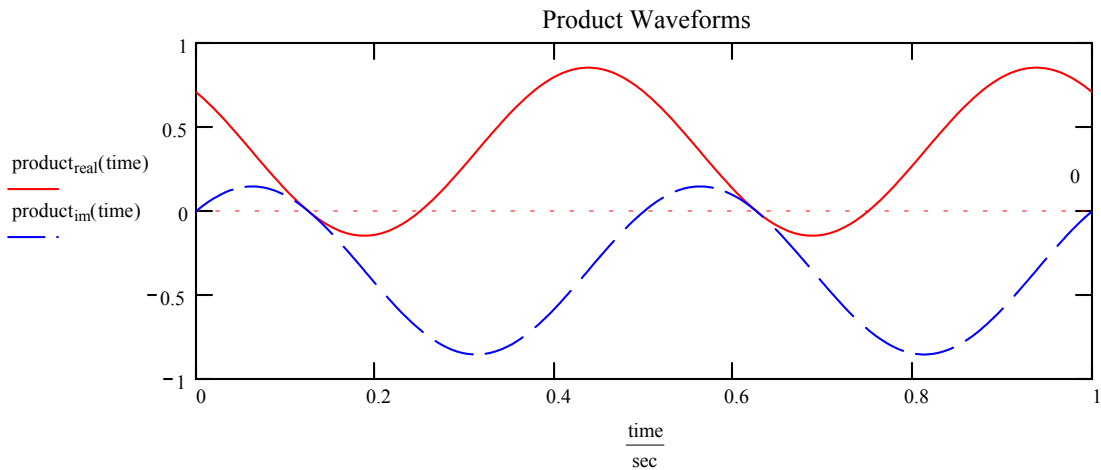
$$\text{test}_{\text{im}}(t) := \sin(2 \cdot \pi \cdot \text{Hz} \cdot t)$$



Next the product wave is produced by multiplying the two waveforms (signal and test)

$$\text{product}_{\text{real}}(t) := \text{test}_{\text{real}}(t) \cdot \text{signal}(t)$$

$$\text{product}_{\text{im}}(t) := \text{test}_{\text{im}}(t) \cdot \text{signal}(t)$$



Integrate the product function to determine the "real" and "imaginary" signal.

Integrate the real signal:

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}_{\text{real}}(t) dt = 0.354 \text{ sec}$$

Integrate the imaginary signal:

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}_{\text{im}}(t) dt = -0.354 \text{ sec}$$

This result is consistent with what you expect. Since the signal is midway between a sine and a cosine wave. The intensity of the real signal is $0.5 \cdot \cos(45\text{-deg}) = 0.354$ and the intensity of the imaginary signal is $0.5 \cdot \sin(-45\text{-deg}) = -0.354$. The negative sign comes from the angle relative to the cosine and sine test waves (the signal is ahead of one and behind the other).

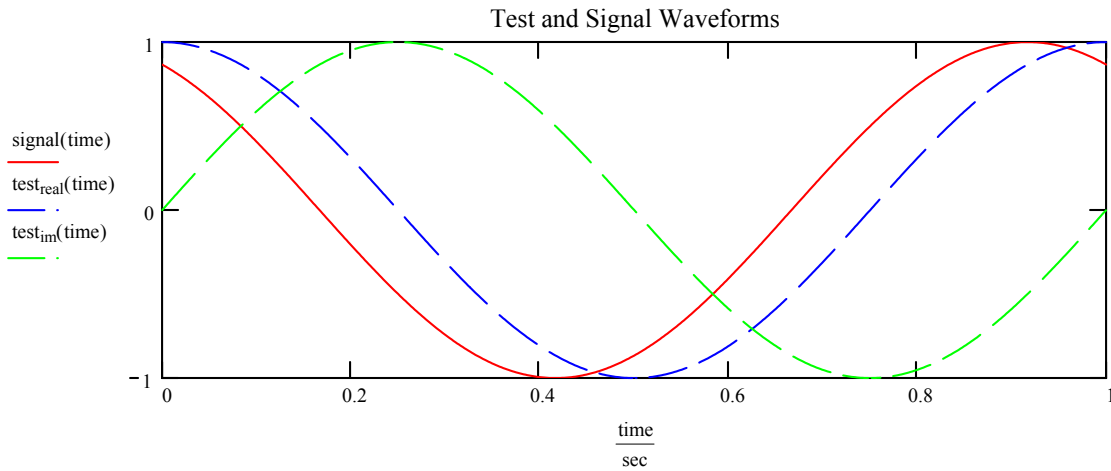
One more angle just to be certain everything fits together.

Generate the waveforms:

$$\text{signal}(t) := \cos\left(2 \cdot \pi \cdot \text{Hz} \cdot t + \frac{\pi}{6}\right) \quad \text{A 30 degree phase angle}$$

$$\text{test}_{\text{real}}(t) := \cos(2 \cdot \pi \cdot \text{Hz} \cdot t)$$

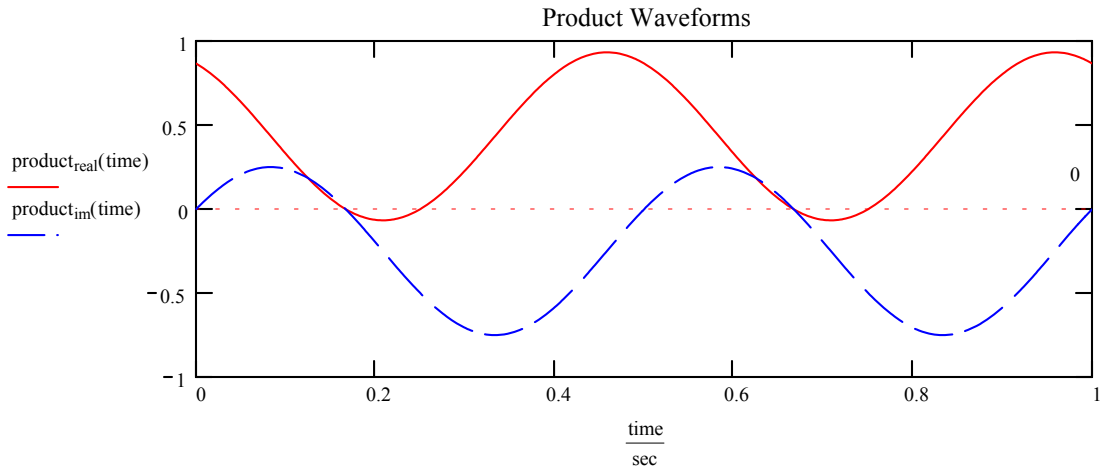
$$\text{test}_{\text{im}}(t) := \sin(2 \cdot \pi \cdot \text{Hz} \cdot t)$$



Next the product wave is produced by multiplying the two waveforms (signal and test)

$$\text{product}_{\text{real}}(t) := \text{test}_{\text{real}}(t) \cdot \text{signal}(t)$$

$$\text{product}_{\text{im}}(t) := \text{test}_{\text{im}}(t) \cdot \text{signal}(t)$$



Integrate the product function to determine the "real" and "imaginary" signal.

Integrate the real signal:

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}_{\text{real}}(t) dt = 0.433 \text{ sec}$$

Integrate the imaginary signal:

$$\int_{0\text{-sec}}^{t_{\text{acquire}}} \text{product}_{\text{im}}(t) dt = -0.25 \text{ sec}$$

This result is consistent with what you expect. Since the signal is midway between a sine and a cosine wave. The intensity of the real signal is $0.5 \cdot \cos(30 \cdot \text{deg}) = 0.433$ and the intensity of the imaginary signal is $0.5 \cdot \sin(-30 \cdot \text{deg}) = -0.25$. The negative sign comes from the angle relative to the cosine and sine test waves (the signal is ahead of one and behind the other).

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